Jacobian elliptic function solutions of the discrete cubic–quintic nonlinear Schrödinger equation

2007 J. Phys. A: Math. Theor. 40 6133

(http://iopscience.iop.org/1751-8121/40/23/009)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 6133–6145 doi:10.1088/1751-8113/40/23/009

Jacobian elliptic function solutions of the discrete cubic–quintic nonlinear Schrödinger equation

G C Latchio Tiofack[1], **Alidou Mohamadou**[1,2] **and T C Kofane**[1,3]

[1] Laboratory of Mechanics, Department of Physics, Faculty of Science, University of Yaoundé I, PO Box 812 Yaoundé Cameroon
[2] Condensed Matter Laboratory, Department of Physics, Faculty of Science, University of Douala, PO Box 24157 Douala Cameroon
[3] The Abdus Salam International Centre for Theoretical Physics, PO Box 586 Strada Costiera, 11, I-34014 Trieste, Italy

Received 21 December 2006, in final form 10 April 2007
Published 22 May 2007
Online at stacks.iop.org/JPhysA/40/6133

Abstract
The study of solitary wave solutions is of prime significance for the nonlinear Schrödinger equation with higher order dispersion and/or higher degree nonlinearities in nonlinear physical systems. We derive the discrete cubic–quintic nonlinear Schrödinger equation from a Hamiltonian using different Poisson brackets. By using the extended Jacobian elliptic function approach, we investigate the abundant exact stationary solitons and periodic waves solution of this equation. These solutions include, Jacobian periodic solutions, alternating phase Jacobi periodic solution, kink and bubble soliton solutions, alternating phase kink soliton solution and alternating phase bubble soliton solution, provided that coefficients are bound by special relation. And then with the aid of symbolic computation, we present in explicit form these solutions. The stability of bubble and kink soliton as well as alternating kink and alternating bubble soliton are also investigated.

PACS numbers: 42.65.−k, 42.65.Sf, 47.54.+r, 05.45.Yv, 42.65.Tg

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Solitons, in general, manifest themselves in a large variety of wave/particle systems in nature: practically in any system that possesses both dispersion (in time or space) and nonlinearity. Solitons have been identified in optics, plasmas, fluids, condensed matter, particle physics, and astrophysics. Over the past decades, the rapid progresses on the fronts of spatial and spatiotemporal optical solitons had been the hotspots [1, 2] and among these areas, the forefront of soliton research has been shifted to optics. Spatial optical solitons are self-trapped optical beams that exist by virtue of the balance between diffraction and nonlinearity [1, 2].

Spatiotemporal optical solitons are nondiffracting and nondispersing wave packets propagating in nonlinear optical media, and in [1], the authors offer an up-to-date survey of experimental and theoretical results in this field. It was recently shown that discrete solitons exist in several physics field [3] and can be used to realize various functional operations, such as blocking, routing, logic functions, time gating, etc. [4].

The investigation of exact solutions, in particular solitons, for nonlinear mathematical physics equations is an important and interesting subject. Those solutions play important roles in understanding the fundamental properties of physical systems [5]. So, searching for some exact physically significant solution is the important topic of the solitons theory. There is a wide range of approaches for finding special solutions of the nonlinear partial differential equations, such as the inverse scattering method [6], the Backlünd transformation [7], the homogeneous balance method [8], the multilinear separation approach [9], the tanh method [10], the standard truncated Painlevé expansion [11, 12], the Jacobian elliptic function method [13] and so on. However, less work has been done to investigate exact solutions of a nonlinear differential–difference equation (NDDE), because to extend the above methods to a NDDE is rather difficult.

One of the prototypical differential–difference model that is both physically relevant is the so-called discrete nonlinear Schrödinger (DNLS) equation. It represents one of the simplest equations in which the combination of dispersive effects with the cubic nonlinearity leads to the localized solutions of soliton type. The most direct implementation of DNLS equations can be identified in one-dimensional array of coupled optical waveguides [14]. Light-induced photonic lattices [15] and an array of Bose–Einstein condensate [16] have recently emerged as an important application of such equations. It is well known that the standard DNLS equation cannot have bright or dark soliton solutions with an arbitrary position relative to the lattice; only site-centred and bound-centred solitons are possible. Physically, this fact is related to the presence of the Peirls–Nabarro barrier, an effective potential periodic with the spacing of the lattice. However, there are some 'exceptional' discretizations which do support families of solitons with arbitrary position relative to the lattice [17]. An important example of this is the integrable Ablowitz–Ladik (AL) equation. The AL system has N-soliton solutions and it is well known that the Peierls–Nabarro barrier vanishes in this case. However, the AL model is not physically realistic because it does not contain the Kerr-nonlinearity. One of the physical effects which extend a Kerr-type nonlinearity is the saturation of the nonlinearity which appears in the model of propagation of optical pulse in various doped fibres [18]. Khare et al have used elliptic-function identities to find exact solutions of DNLS equation with saturable nonlinearity [19]. More recently, Dmitriev *et al* [20] presented a class of DNLS equations for general polynomial nonlinearity and obtained soliton solutions in the case of cubic nonlinearity. For the continuous NLS equation with attractive interaction it is well known that the higher order nonlinearities (higher than the cubic) lead to the collapse in a finite time (blow up) if the norm exceeds a critical value, even in the one-dimensional case. The interplay between dimensionality and the order of nonlinearity has indeed been used in the past as a way to investigate collapse in low-dimensional nonlinear systems [21]. Although in a DNLS system true collapse cannot occur, due to the conservation norm, it may be possible that some of the features observed in the continuous NLS system about localized solutions may also exist at the discrete level. In particular, it is known that the 1D continuous NLS equation with high-order nonlinearity (e.g., quintic) there exists only one localized solution for each value of the norm (critical norm), the so-called Townes soliton [22], which separates collapsing and decaying solution while being marginally stable against decay or collapse. In the presence of an external field, for example a periodic potential, it is possible to stabilize such solutions of continuous NLS with higher order nonlinearities against decay, extending the existence range of localized solutions from a single value of the norm to a whole interval. Since the discrete NLS equation can be viewed as a tight binding model of the continuous NLS with a periodic potential, it is of interest to investigate the existence of discrete, stable localized solutions when higher order nonlinearity are introduced in the DNLS equation.

The paper is structured as follows. In section 2, we derive the discrete cubic–quintic nonlinear Schrödinger (DCQNLS) equation. Stationary solitons and periodic waves solution are found in section 3. Then, in section 3.6 we study the stability of dark solitons: bubble and kink solitons solution. Finally, section 4 concludes the paper.

2. The model

A class of DNLS equations with arbitrarily high-order nonlinearities was introduced in several works [23]:

$$\mathrm{i}\dot{\psi}_n + (\psi_{n+1} + \psi_{n-1})[1 + f(|\psi_n|^2)] - \nu f'(|\psi_n|^2)\psi_n = 0 \tag{1}$$

where the function $f(x)$, a polynomial of degree $p+1$, is given by:

$$f(x) = \sum_{j=0}^{p} \alpha_j x^{j+1}. \tag{2}$$

This equation is derived from the Hamiltonian given by

$$H = \sum_{n=1}^{N} [|\psi_n - \psi_{n+1}|^2 - 2|\psi_n|^2 + \nu \ln(1 + f(|\psi_n|^2))] \tag{3}$$

using the deformed Poisson brackets

$$\{\psi_n, \psi_m^*\} = \mathrm{i}(1 + f(|\psi_n|^2))\delta_{nm}, \qquad \{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = 0. \tag{4}$$

The Poisson brackets can be compactly written as

$$\{U, V\} = \mathrm{i}\sum_{n=1}^{N} \left(\frac{\mathrm{d}U}{\mathrm{d}\psi_n}\frac{\mathrm{d}V}{\mathrm{d}\psi_n^*} - \frac{\mathrm{d}V}{\mathrm{d}\psi_n}\frac{\mathrm{d}U}{\mathrm{d}\psi_n^*}\right)[1 + f(|\psi_n|^2)]. \tag{5}$$

The equation of motion is

$$\dot{\psi}_n = \{H, \psi_n\}. \tag{6}$$

In the particular case of the cubic–quintic nonlinearity, the function f can be written as:

$$f(|\psi_n|^2) = \alpha_0|\psi_n|^2 + \alpha_1|\psi_n|^4. \tag{7}$$

Substituting equation (4) into equation (3), we obtain the DCQNLS equation

$$\begin{aligned} &\mathrm{i}\dot{\psi}_n + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n + \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \\ &\qquad + \eta|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) = 0, \end{aligned} \tag{8}$$

with the normalized coefficients $\alpha = 1, \nu\alpha_0 = -2,\ \beta = -2\nu\alpha_1,\ \gamma = \alpha_0,\ \eta = \alpha_1$. This equation describes the propagation of discrete self-trapped beams in an array of weakly coupled nonlinear optical waveguides. In equation (8), ψ_n is a complex-valued 'wavefunction' at site n. The coupling constant is α, while β and γ are cubic nonlinearities and η is the quintic nonlinearity. The DNLS equation appears ubiquitously [1] throughout modern science since it represents one of the simplest equations in which the combination of dispersive effects with a cubic nonlinearity leads to localized solutions of solitons type. Most notably is the role it plays in understanding the propagation of the electromagnetic wave in glass fibres and other optical waveguides [24]. More recently the DNLS equation has been used as a tight binding model for Bose–Einstein condensates in optical lattices [25, 26]. From a physical point of view, it is of interest to study the effects of including high order nonlinear terms (higher than cubic) in the equation on discrete solitons. These terms appear in different physical contexts such as Bose gases with hard core interactions in the Tonk–Giradeau regime [27] and low-dimensional Bose–Einstein condensate in which quintic nonlinearities in the NLS equation are used to model three-body interactions [28]. A self-focusing cubic–quintic NLS equation is also used in nonlinear optics as a model for photonic crystals [29].

3. Exact solutions of the QDNLS equation

In order to obtain some exact solutions of equation (8), we use the Jacobian elliptic function approach [13]. Firstly, we make the transformations

$$\psi_n = \mathrm{e}^{\mathrm{i}\theta_n}\phi_n(\xi_n), \qquad \theta_n = pn + \omega t + \theta_0, \qquad \xi_n = kn + ct + \zeta_0, \tag{9}$$

and

$$\psi_{n+1} = \mathrm{e}^{\mathrm{i}\theta_n}\mathrm{e}^{\mathrm{i}p}\phi_{n+1}(\xi_n), \qquad \psi_{n-1} = \mathrm{e}^{\mathrm{i}\theta_n}\mathrm{e}^{-\mathrm{i}p}\phi_{n-1}(\xi_n). \tag{10}$$

With the expression $\mathrm{e}^{\pm\mathrm{i}p} = \cos(p) \pm \mathrm{i}\sin(p)$, equation (8) is therefore reduced to

$$\begin{aligned}-\omega\phi_n &+ \mathrm{i}c\phi_n' + \alpha[\cos(p)(\phi_{n+1}+\phi_{n-1}) + \mathrm{i}\sin(p)(\phi_{n+1}-\phi_{n-1}) - 2\phi_n] + \beta\phi_n^3\\ &+ \gamma\phi_n^2[\cos(p)(\phi_{n+1}+\phi_{n-1}) + \mathrm{i}\sin(p)(\phi_{n+1}-\phi_{n-1})]\\ &+ \eta\phi_n^4[\cos(p)(\phi_{n+1}+\phi_{n-1}) + \mathrm{i}\sin(p)(\phi_{n+1}-\phi_{n-1})] = 0\end{aligned} \tag{11}$$

Separating the real and imaginary parts, one gets

$$-(\omega+2\alpha)\phi_n + \beta\phi_n^3 + \cos(p)\left(\alpha+\gamma\phi_n^2+\eta\phi_n^4\right)(\phi_{n+1}+\phi_{n-1}) = 0, \tag{12a}$$

$$c\phi_n' + \sin(p)\left(\alpha+\gamma\phi_n^2+\eta\phi_n^4\right)(\phi_{n+1}-\phi_{n-1}) = 0. \tag{12b}$$

We use the following series expansion as a solution of equations (12*a*) and (12*b*):

$$\phi_n(\xi_n) = a_0 + a_1 sn(\xi_n) + a_2 sn^2(\xi_n), \tag{13}$$

where $sn(\xi_n) \equiv sn(\xi_n, m)$, $m(0 < m < 1)$ is a modulus of Jacobian elliptic functions and $a_i\ \ (i = 0,\ 1,\ 2)$ are constants to be determined. From the identity

$$sn(\xi_1+\xi_2) = \frac{sn(\xi_1)cn(\xi_2)dn(\xi_2) + sn(\xi_2)cn(\xi_1)dn(\xi_1)}{1 - m^2sn^2(\xi_1)sn^2(\xi_2)}, \tag{14}$$

we obtain

$$\begin{aligned}\phi_{n+1}(\xi_n) = a_0 &+ a_1\left[\frac{sn(\xi_n)cn(k)dn(k) + sn(k)cn(\xi_n)dn(\xi_n)}{1-m^2sn^2(\xi_n)sn^2(k)}\right]\\ &+ a_2\left[\frac{sn(\xi_n)cn(k)dn(k) + sn(k)cn(\xi_n)dn(\xi_n)}{1-m^2sn^2(\xi_n)sn^2(k)}\right]^2,\end{aligned} \tag{15a}$$

$$\begin{aligned}\phi_{n-1}(\xi_n) = a_0 &+ a_1\left[\frac{sn(\xi_n)cn(k)dn(k) - sn(k)cn(\xi_n)dn(\xi_n)}{1-m^2sn^2(\xi_n)sn^2(k)}\right]\\ &+ a_2\left[\frac{sn(\xi_n)cn(k)dn(k) - sn(k)cn(\xi_n)dn(\xi_n)}{1-m^2sn^2(\xi_n)sn^2(k)}\right]^2.\end{aligned} \tag{15b}$$

Substituting equations (13) and (15) into equation (12), clearing the denominator and setting the coefficients of all power like $sn^i(\xi_n)$ $(i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ and $cn(\xi_n)dn(\xi_n)sn^j(\xi_n)$ $(j = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ to zero, one can obtain a series of over-determined algebraic equations, which are omitted to avoid the tediousness. By solving these equations and according to equations (9) and (13), we obtain the following solutions of equation (8).

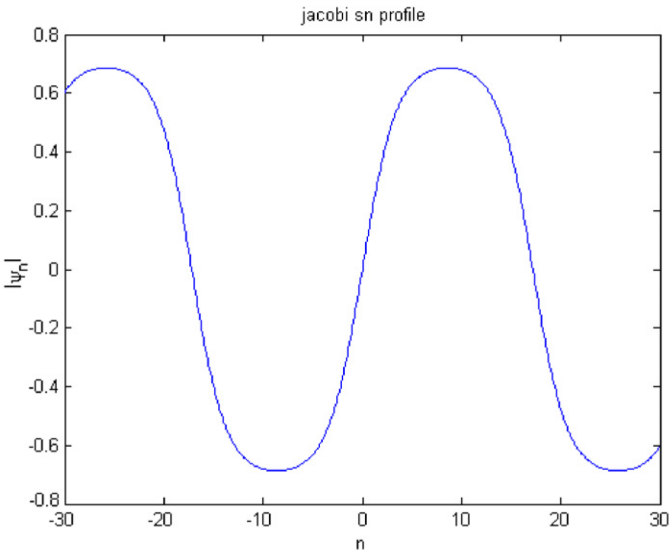


Figure 1. Profile of Jacobian elliptic *sn* function solution $\psi_{n,1}$ $m = 0.9,\ \ p = 0,\ \ \alpha = -8, k = 0.3,$ $\gamma = 1,\ \ \delta = 0.025,\ \ \theta_0 = 0,\ \ \xi_0 = 0$ at $t = 0$.

3.1. Jacobi sn function solution

$$\psi_{n,1} = \pm \mathrm{msn}(k)\sqrt{\frac{-\gamma \pm \sqrt{\gamma^2 - 4\eta\alpha}}{2\eta}} sn(kn + \zeta_0) \exp[\mathrm{i}(-2\alpha + 2\alpha cn(k)dn(k))t + \mathrm{i}\theta_0]. \quad (16)$$

3.2. Jacobi cn function solution

$$\psi_{n,2} = \pm m^2 sn^2(k)\sqrt{\frac{\alpha}{\gamma}}(1 - cn^2(kn + \zeta_0)) \exp[\mathrm{i}(-2\alpha + 2\alpha cn^2(k)dn^2(k))t + \mathrm{i}\theta_0]. \quad (17)$$

The properties of solutions (16) and (17), are shown in figures 1 and 2, respectively.

3.3. Alternating phase Jacobi sn function solution

$$\psi_{n,3} = \pm(-1)^n msn(k)\sqrt{\frac{-\gamma \pm \sqrt{\gamma^2 - 4\eta\alpha}}{2\eta}} sn(kn + \zeta_0) \exp[\mathrm{i}(-2\alpha - 2\alpha cn(k)dn(k))t + \mathrm{i}\theta_0]. \quad (18)$$

3.4. Alternating phase Jacobi cn function solution

$$\psi_{n,4} = \pm(-1)^n m^2 sn^2(k)\sqrt{\frac{\alpha}{\gamma}}(1 - cn^2(kn + \zeta_0)) \exp[\mathrm{i}(-2\alpha - 2\alpha cn^2(k)dn^2(k))t + \mathrm{i}\theta_0]. \quad (19)$$

The properties of solutions (18) and (19), are shown in figures 3 and 4, respectively. When $m \to 1$, we obtain the following new stationary solutions.

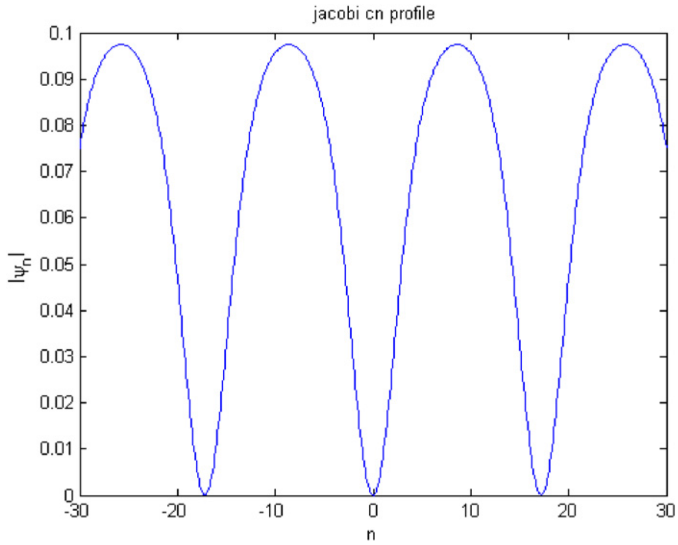


Figure 2. Profile of Jacobian elliptic *cn* function $\psi_{n,2}$ $m = 0.9,\ p = 0,\ \alpha = 2,\ k = 0.3,\ \gamma = 1,\ \theta_0 = 0,\ \xi_0 = 0$ at $t = 0$.

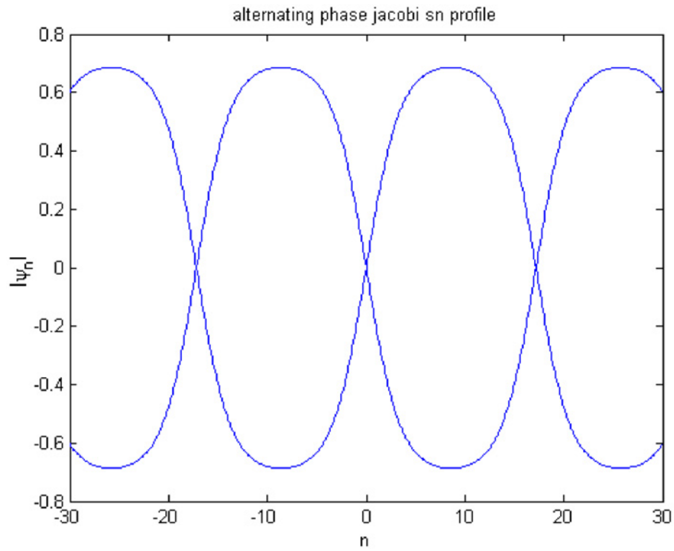


Figure 3. Profile of alternating phase Jacobian elliptic sn function solution $\psi_{n,3}$ $m = 0.9,\ p = \pi,\ \alpha = -8,\ k = 0.3,\ \gamma = 1,\ \delta = 0.025,\ \theta_0 = 0,\ \xi_0 = 0$ at $t = 0$.

3.5. Kink soliton solution

$$\psi_{n,5} = \pm\sqrt{\frac{-\gamma \pm \sqrt{\gamma^2 - 4\eta\alpha}}{2\eta}} \tanh(k) \tanh(kn + \zeta_0) \exp[\mathrm{i}(-2\alpha + 2\alpha \sec h^2(k))t + \mathrm{i}\theta_0]. \quad (20)$$

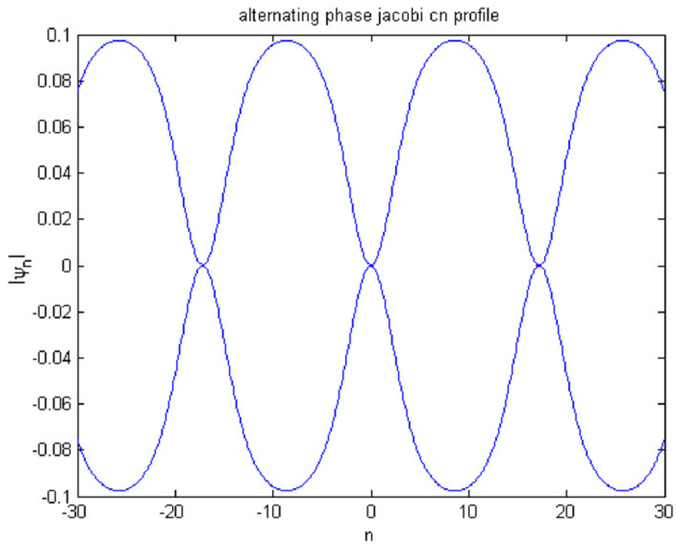


Figure 4. Profile of alternating phase Jacobian elliptic cn function solution $\psi_{n,4}$ $m = 0.9,\ p = \pi,\ \alpha = 2,\ k = 0.3,\ \gamma = 1,\ \theta_0 = 0,\ \xi_0 = 0$ at $t = 0$.

3.6. Bubble soliton solution

$$\psi_{n,6} = \pm\sqrt{\frac{\alpha}{\gamma}}\tanh^2(k)\tanh^2(kn+\zeta_0)\exp[\mathrm{i}(-2\alpha + 2\alpha\sec h^4(k))t + \mathrm{i}\theta_0]. \tag{21}$$

The static dark solitons of the NLS equation can be classified under two broad classes. Bubbles are one-, two- and three-dimensional nontopological solitons arising typically in models with competing interactions [30–32]. The second class includes topological solitons of the Gross–Pitayevski equation and their one-dimensional counterparts, kinks. The static bubbles are always unstable [30–34], and this property endows them with a transparent physical interpretation as nuclei of the first-order transition [35]. The properties of the solutions (20) and (21) are shown in figures 5 and 6, respectively.

3.7. Alternating phase kink soliton solution

$$\psi_{n,7} = \pm(-1)^n\sqrt{\frac{-\gamma \pm \sqrt{\gamma^2 - 4\eta\alpha}}{2\eta}}\tanh(k)\tanh(kn+\zeta_0)$$
$$\times \exp[\mathrm{i}(-2\alpha - 2\alpha\sec h^2(k))t + \mathrm{i}\theta_0]. \tag{22}$$

3.8. Alternating phase bubble soliton solution

$$\psi_{n,8} = \pm(-1)^n\sqrt{\frac{\alpha}{\gamma}}\tanh^2(k)\tanh^2(kn+\zeta_0)\exp[\mathrm{i}(-2\alpha - 2\alpha\sec h^4(k))t + \mathrm{i}\theta_0] \tag{23}$$

The properties of the solutions (22) and (23) are shown in figures 7 and 8, respectively. Here, $k,\ \zeta_0,\ \theta_0$ are arbitrary constants. These solutions exist if the quantity under square root is positive. Thus, equations (16), (18), (20) and (22) require that $\gamma^2 - 4\eta\alpha \geqslant 0$, $\eta\alpha < 0$ if $\gamma > 0$, $\eta\alpha > 0$ if $\gamma < 0$ or $\gamma^2 - 4\eta\alpha \geqslant 0$, $\eta\alpha > 0$ if $\gamma < 0$, $\alpha < 0$ if $\gamma > 0$. Equations (17), (19), (21) and (23) require $\alpha\gamma > 0$.

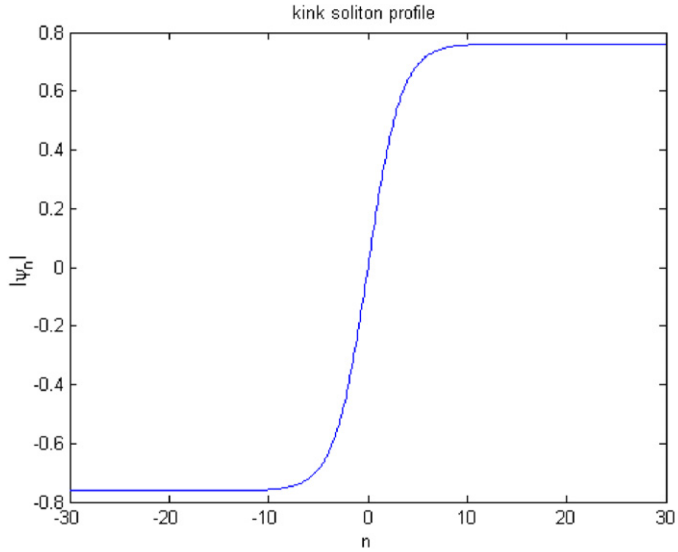


Figure 5. Profile of kink soliton solution $\psi_{n,5}$ $m = 1$, $p = 0$, $\alpha = -8$, $k = 0.3$, $\gamma = 1$, $\delta = 0.025$, $\theta_0 = 0$, $\xi_0 = 0$ at $t = 0$.

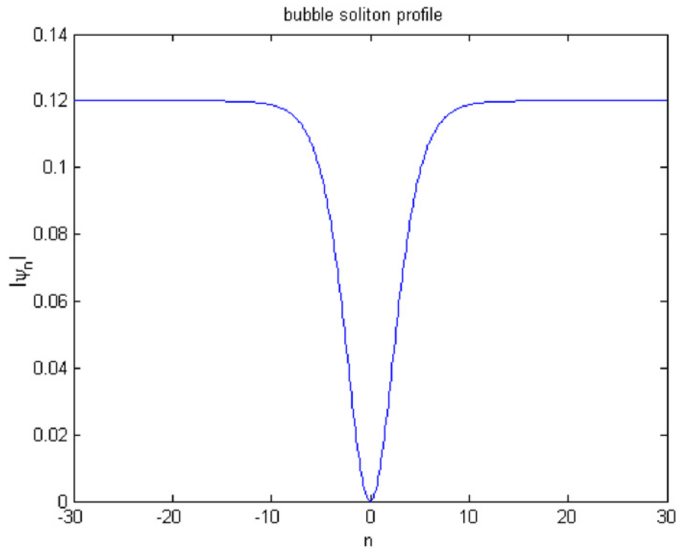


Figure 6. Profile of bubble soliton solution $\psi_{n,6}$ $m = 1$, $p = 0$, $\alpha = 2$, $k = 0.3$, $\gamma = 1$, $\theta_0 = 0$, $\xi_0 = 0$ at $t = 0$.

The approximate solutions (16)–(23) contain the free parameter ζ_0 defining the soliton position. However, it is well known that the standard DNLS equation have stationary soliton solutions only for a discrete set of values of ζ_0 (e.g. on-site, $\zeta_0 = 0$, and inter-site, $\zeta_0 = \frac{1}{2}$). However, there are several exceptional discretizations of the cubic NLS equation where stationary soliton solutions exist for any ζ_0, or, in other words, they can be placed anywhere with respect to the lattice; otherwise put, the Peirls–Nabarro potential absent for stationary solutions of these models. The DCQNLS model used in this work is then one of the examples in which the stationary solutions include an arbitrary translational invariance.

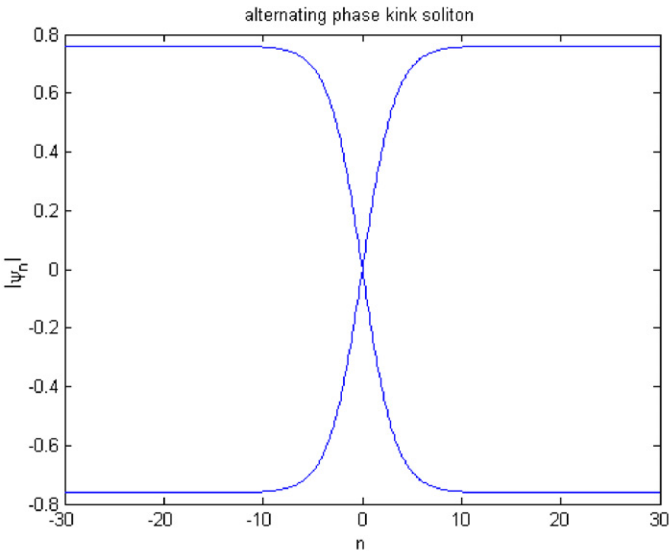


Figure 7. Profile of alternating phase kink soliton solution $\psi_{n,7}$ $m = 1$, $p = \pi$, $\alpha = -8$, $k = 0.3$, $\gamma = 1$, $\delta = 0.025$, $\theta_0 = 0$, $\xi_0 = 0$ at $t = 0$.

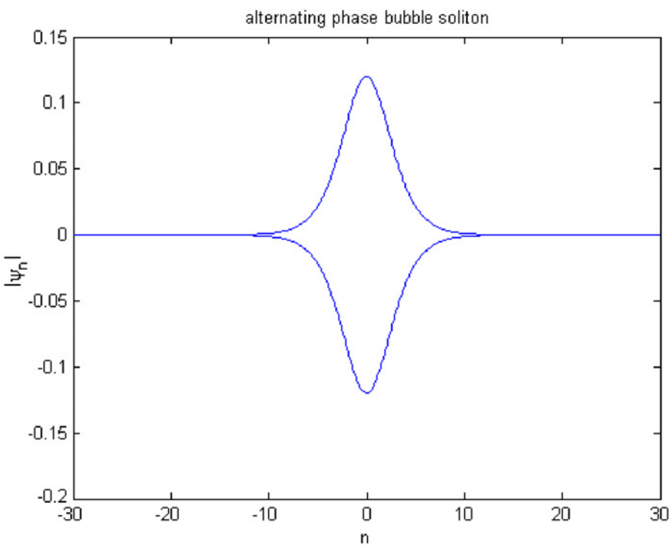


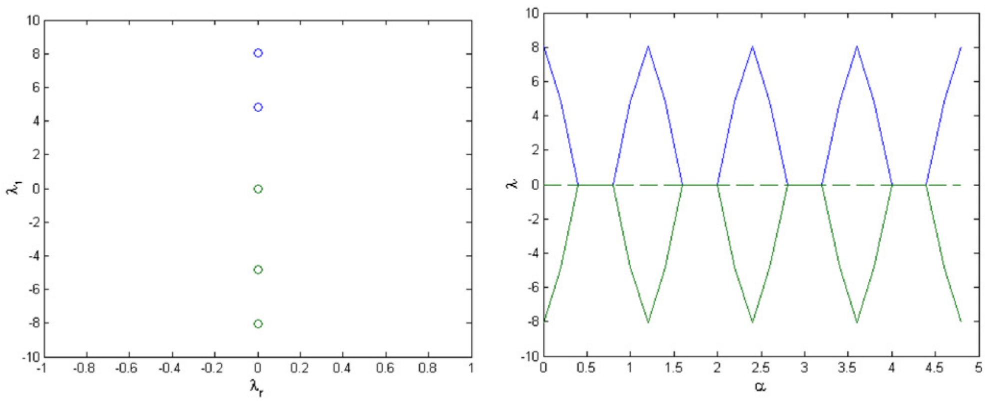
Figure 8. Profile of alternating phase bubble soliton solution $\psi_{n,8}$ $m = 1$, $p = \pi$, $\alpha = 2$, $k = 0.3$, $\gamma = 1$, $\theta_0 = 0$, $\xi_0 = 0$ at $t = 0$.

The continuous bubble are always unstable [35], it would then interesting to investigate whether discreteness effects could stabilize bubbles.

4. Stability analysis

In order to study the linear stability of the exact solutions $\psi_{n,j}$ obtained in section 3, we introduce the following expansion

$$\psi_n(t) = (\phi_{n,j} + \varepsilon_n(t))\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{24}$$



(*a*) (*b*)

Figure 9. (*a*) Eigenvalue spectrum of the bubble soliton solution $\psi_{n,6}$ $p = 0,\ k = 0.3,\ \beta = 0.4,\ \gamma = 1,\ \eta = -1,\ \theta_0 = 0.$ (*b*) Spectral plane of the bubble soliton solution $\psi_{n,6}$ $p = 0,\ k = 0.3,\ \beta = 0.4,\ \gamma = 1,\ \eta = -1,\ \theta_0 = 0.$

applied in the frame rotating with frequency ω of the solution. Substituting equation (24) into the DCQNLS equation, we find that the linearized equation satisfied by $\varepsilon_n(t)$ is given by

$$\begin{aligned}
\mathrm{i}\dot{\varepsilon_n} &- \omega\varepsilon_n + \alpha(\varepsilon_{n+1}\mathrm{e}^{\mathrm{i}p} + \varepsilon_{n-1}\mathrm{e}^{-\mathrm{i}p} - 2\varepsilon_n) + \beta(2\varepsilon_n + \varepsilon_n^*)\phi_n^2 \\
&+ \gamma\left[\phi_n^2(\varepsilon_{n+1}\mathrm{e}^{\mathrm{i}p} + \varepsilon_{n-1}\mathrm{e}^{-\mathrm{i}p}) + \phi_n(\varepsilon_n + \varepsilon_n^*)(\phi_{n+1}\mathrm{e}^{\mathrm{i}p} + \phi_{n-1}\mathrm{e}^{-\mathrm{i}p})\right] \\
&+ \eta\left[\phi_n^4(\varepsilon_{n+1}\mathrm{e}^{\mathrm{i}p} + \varepsilon_{n-1}\mathrm{e}^{-\mathrm{i}p}) + 2\phi_n^3(\varepsilon_n + \varepsilon_n^*)(\phi_{n+1}\mathrm{e}^{\mathrm{i}p} + \phi_{n-1}\mathrm{e}^{-\mathrm{i}p})\right] = 0. \qquad (25)
\end{aligned}$$

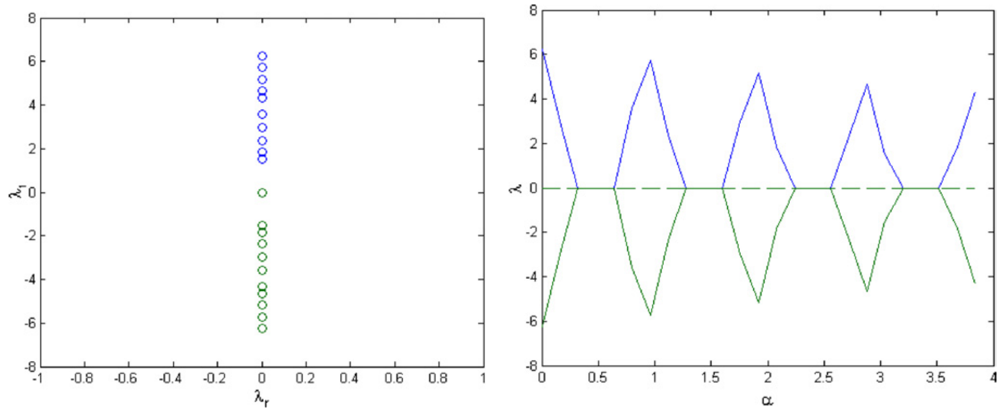
Expanding $\varepsilon_n(t)$ in real and imaginary parts: $\varepsilon_n(t) = a_n(t) + \mathrm{i}b_n(t)$, the linearized equations can be written as

$$\begin{pmatrix}\{a_n\}\\ \{b_n\}\end{pmatrix} = \begin{pmatrix} M_1 & M_3\\ M_2 & M_4\end{pmatrix}\begin{pmatrix}\{a_n\}\\ \{b_n\}\end{pmatrix} \equiv \hat{M}\begin{pmatrix}\{a_n\}\\ \{b_n\}\end{pmatrix}, \qquad (26)$$

with:

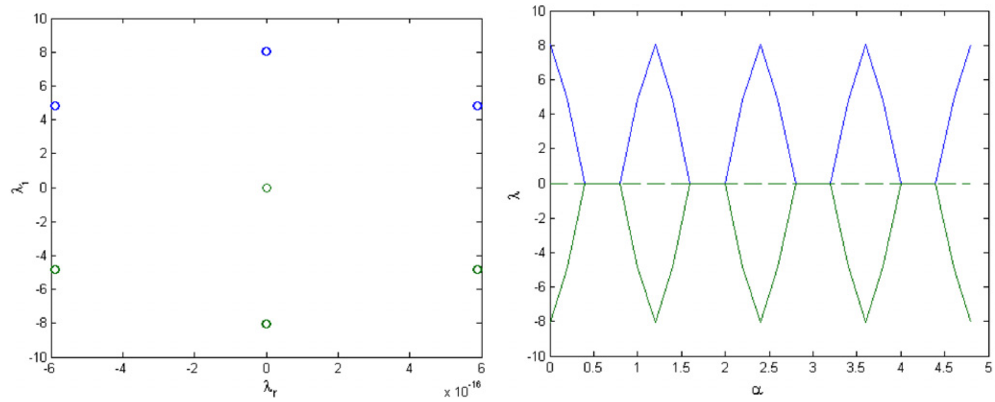
$$\begin{cases}
(M_1)_{nm} = -\sin(p)(\delta_{n,m+1} - \delta_{n,m-1})\left(\alpha + \gamma\phi_n^2 + \eta\phi_n^4\right) \\
\qquad\quad - 2\sin(p)\phi_n(\phi_{n+1} - \phi_{n-1})\left(\gamma + 2\eta\phi_n^2\right)\delta_{n,m}, \\
(M_2)_{nm} = -\left(\omega + 2\alpha - 3\beta\phi_n^2\right)\delta_{n,m} + \cos(p)(\delta_{n,m+1} + \delta_{n,m-1})\left(\alpha + \gamma\phi_n^2 + \eta\phi_n^4\right) \\
\qquad\quad + 2\cos(p)\phi_n(\phi_{n+1} + \phi_{n-1})\left(\gamma + 2\eta\phi_n^2\right)\delta_{n,m}, \\
(M_3)_{nm} = \left(\omega + 2\alpha - \beta\phi_n^2\right)\delta_{n,m} - \cos(p)(\delta_{n,m+1} + \delta_{n,m-1})\left(\alpha + \gamma\phi_n^2 + \eta\phi_n^4\right), \\
(M_4)_{nm} = -\sin(p)(\delta_{n,m+1} - \delta_{n,m-1})\left(\alpha + \gamma\phi_n^2 + \eta\phi_n^4\right),
\end{cases} \qquad (27)$$

Stationary soliton solution is linearly stable if and only if the matrix $\hat{M}$ has all its eigenvalues on the imaginary axis. Otherwise, the solution is unstable. In our stability analysis, we have used periodic boundary conditions. The eigenvalue spectrum of the matrix $\hat{M}$ determines the stability of the exact solutions bubble and kink soliton (see figures 9 and 10, respectively). The eigenvalue spectrum always contains eigenvalues which are zero. This eigenvalue corresponds to the translational invariance (ζ_0) and to the invariance of the solution $\psi_{n,j}$ to a constant phase factor, respectively. Since the spectra plane (λ_r, λ_i) of these solutions shows that all the eigenvalues are in the imaginary axis, one can say that both kink and bubble solutions are stable. Figures 11 and 12 depict the eigenvalues spectrum of alternating bubble and alternating



(*a*) (*b*)

Figure 10. (*a*) Eigenvalue spectrum of the kink soliton solution $\psi_{n,5}$ $p = 0,\ k = 0.3,\ \beta = 0.4,\ \gamma = -1,\ \eta = 0.025,\ \theta_0 = 0$. (*b*) Spectral plane of the kink soliton solution $\psi_{n,5}$ $p = 0,\ k = 0.3,\ \beta = 0.4,\ \gamma = -1,\ \eta = 0.025,\ \theta_0 = 0$.



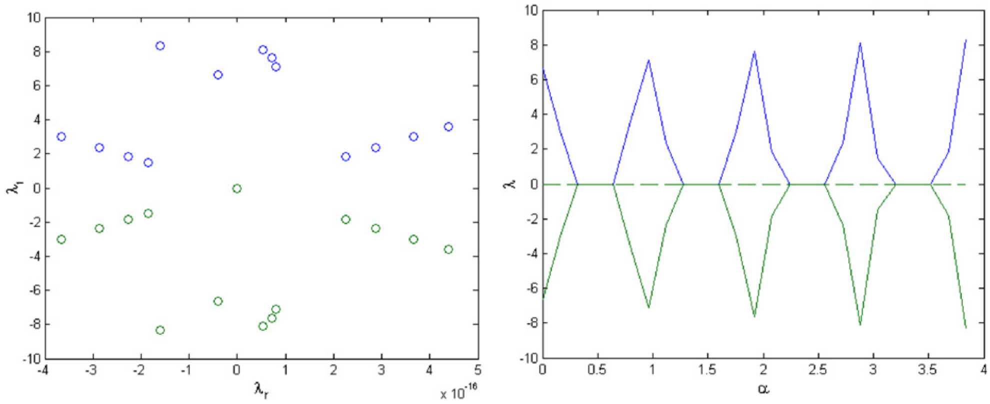
(*a*) (*b*)

Figure 11. (*a*) Eigenvalue spectrum of the alternating bubble soliton solution $\psi_{n,8}$ $p = \pi,\ k = 0.3,\ \beta = 0.4,\ \gamma = 1,\ \eta = -1,\ \theta_0 = 0$. (*b*) Spectral plane of the alternating bubble soliton solution $\psi_{n,8}$ $p = \pi,\ k = 0.3,\ \beta = 0.4,\ \gamma = 1,\ \eta = -1,\ \theta_0 = 0$.

kink soliton solutions. From these figures, we see that complex conjugate eigenvalues leave the imaginary axis and go out in the complex plane. Thus, the alternating bubble and alternating kink solutions are always unstable.

5. Summary and discussion

By using the Jacobian elliptic function approach, we have analysed exact solutions of the DCQNLS equation. The set of solution includes Jacobian periodic solutions, alternating phase Jacobi periodic solution, kink and bubble soliton solutions, alternating phase kink soliton solution and alternating phase bubble soliton solution. On the other hand, it happens that solitons admit translational invariance in the same way as in the integrable Ablowitz–Ladik



(*a*) (*b*)

Figure 12. (*a*) Eigenvalue spectrum of the alternating kink soliton solution $\psi_{n,7}$ $p = \pi,\ k = 0.3,\ \beta = 0.4,\ \gamma = -1,\ \eta = 0.025,\ \theta_0 = 0$. (*b*) Spectral plane of the alternating kink soliton solution $\psi_{n,7}$ $p = \pi,\ k = 0.3,\ \beta = 0.4,\ \gamma = -1,\ \eta = 0.025,\ \theta_0 = 0$.

lattice. We also studied the stability of the stationary solutions under small perturbation and have found that the dark soliton solutions (kink and bubble) are stable, while the alternating phase dark soliton solutions are unstable. Our solutions and related properties are likely to be useful in many physical contexts including nonlinear optics, atomic physics and Bose–Einstein condensate. Clearly, these localized solutions only represent a small subset of large variety of possible solutions admitted by the DCQNLS equation. In forthcoming studies, it would be interesting, for example, to extend the present approach to higher dimensional systems and a set of coupled equations and also investigate exact explicit bright soliton solution of the DCQNLS equation.

References

[1] Malomed B A, Mihalache D, Wise F and Torner J 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** R53
[2] Stegeman G I A, Christodoulides D N and Segev M 2000 *IEEE J. Sel. Top. Quantum Electron* **6** 1419
[3] Kevrekidis P G, Rasmuessen K O and Bishop A R 2001 *Int. J. Mod. Phys.* B **15** 2833
[4] Christodoulides D N and Eugenevia E D 2001 *Opt. Lett.* **26** 1876
Christodoulides D N and Eugenevia E D 2001 *Phys. Rev. Lett.* **87** 233901
[5] Cardner C S, Kruskal J M and Muira R M 1967 *Phys. Rev. Lett.* **19** 1095
[6] Wahlquist H D and Estabrook F B 1971 *Phys. Lett.* **31** 1386
[7] Wang M L 1995 *Phys. Lett.* A **199** 169
[8] Tang X Y, Lou S Y and Zhang Y 2002 *Phys. Rev.* E **66** 046601
[9] Fran E G 2001 *Phys. Lett.* A **282** 18
[10] Pickerng A 1993 *J. Phys. A: Math. Gen.* **26** 4395
[11] Yomba E and Kofané T C 1999 *Physica* D **125** 105
[12] Liu S K, Fu Z T, Liu S D and Zhao Q 2001 *Phys. Lett.* A **289** 69
[13] Dai C and Zhang J 2006 *Chaos, Solitons Fractals* **27** 1042
[14] Eisenberg H S, Morandotti R, Silberberg Y, Arnold J M, Pennelli G and Aitchison J S 2002 *J. Opt. Soc. Am.* B **19** 2938–44
[15] Sukhorukov A A, Kivshar Y S, Eisenberg H S and Silberberg Y 2003 *IEEE J. Quantum Electron* **39** 31–50
[16] Cataliotti F S, Fallani L, Ferlaino F, Fort C, Maddaloni M and Inguscio 2003 *New J. Phys.* **5** 71
[17] Oxtoby O F, Pelinovsky D E and Barashenkov I V 2006 *Nonlinearity* **19** 217
Barashenkov I V, Oxtoby O F and Pelinovsky D E 2005 *Phys. Rev.* E **72** 035602
[18] Coutaz J-L and Kull M 1991 *J. Opt. Soc. Am.* B **8** 95

[19] Khare A, Rasmussen K O, Samuelsen M R and Saxena A 2005 *J. Phys. A: Math. Gen.* **38** 807–14
[20] Dmitriev S V, Kevrekidis P G, Sukhorukov A A, Yoshikawa N and Takeno S 2006 *Preprint* nlin.PS/0603047 v2
[21] Berge L 1998 *Phys. Rep.* **303** 259
[22] Chiao R I, Garmire E and Townes C H 1964 *Phys. Rev. Lett.* **13** 479
[23] Khare A, Rasmussen K O, Samuelsen M R and Saxena A 2006 *Preprint* nlin.PS/0603034 v2
[24] Eisenberg H S, Silberberg Y, Morandotti R, Boyd A R G and Aitchison J S 1998 *Phys. Rev. Lett.* **81** 383
[25] Trombettoni A and Smerzi A 2001 *Phys. Rev. Lett.* **86** 2353
[26] Abdullaev F Kh, Baizakov B B, Darmanyan S A, Konotop V V and Salerno M 2001 *Phys. Rev.* A **64** 043606
[27] Minguzzi A, Vignolo P, Chiofalo and Tosi M P 2001 *Phys. Rev.* A **64** 03605
Damski B 2004 *J. Phys. B: At. Mol. OPT. Phys.* **37** L85
[28] Abdullaev F Kh and Salerno M 2005 *Phys. Rev.* A **72** 033617
[29] Maruno K, Ohta Y and Joshi N 2003 *Phys. Lett.* A **311** 214
[30] Barashenkov I V and Makhankov V G 1988 *Phys. Lett.* A **128** 52
[31] Barashenkov I V, Gocheva A D, Makhankov V G and Puzynin I V 1988 *Physica (Amsterdam)* **34D** 240
[32] Barashenkov I V and Woodford S R 2005 *Phys. Rev.* E **71** 026613
[33] De Bouard A 1995 *SIAM. J. Math. Anal.* **26** 566
[34] Barashenkov I V 1996 *Phys. Rev. Lett.* **77** 1193
[35] Barashenkov I V and Panova Y E 1993 *Physica* D **69** 114